SUPERSONIC EFFLUX OF A GAS FROM A NOZZLE INTO A REGION OF LOW PRESSURE

(SVERKHZVUKOVOYE ISTECHENIYE GAZA IZ COPLA V OBLAST' PONIZHENNOGO DAVLENIYA)

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We consider a plane supersonic jet of ideal gas, flowing out of a nozzle into a region of low pressure with uniformly distributed velocity at the exit.

The theoretical study of such a gas flow, with the additional condition that the flow is steady and irrotational, was originally carried out by Prandtl [1]. Using the method of small disturbances Prandtl confirmed the experimental result that the jet has a periodic structure for small drops in the pressure.

Then, Prandtl's solution was improved in a number of articles. In most cases approximate solutions were obtained having periodic structures and they did not contain either the surface of discontinuity, or the singularities which lead to the necessary introduction of such surfaces. In particular, in [2], continuous periodic solutions were obtained by a method analogous to that of Khristianovich [3].

Yet the literature suggested that more exact solutions must contain the surface of discontinuity, otherwise at some distance from the exit of the nozzle there arises a limit line [4]. The analytical proof of this assertion can be found in [5]. Through the method of Lin [6] it was shown there that under the condition of a sufficiently small drop in pressure there does appear in the jet a limit line as the envelope of the straight-line characteristics, which converge before they can reach the free surface. The flow in such a jet, clearly, must have an aperiodic character.

We remark, at this point, that for the calculated case of supersonic efflux the impossibility of continuous flow at a sufficiently large distance from the center of the nozzle and the aperiodicity of the jet were established in [7].

In the following article the problem of Prandtl on the supersonic efflux of gas from a plane nozzle into a region of lower pressure will be solved by the method presented in [8]. Problems connected with the disappearance of shock waves in the jet are considered.

1. We take the equations for the velocity potential ϕ and the stream function ψ to be

$$\frac{\partial \varphi}{\partial \xi} = -\sqrt{K_1}\,\frac{\partial \psi}{\partial \xi}, \qquad \frac{\partial \varphi}{\partial \eta} = \sqrt{K_1}\,\frac{\partial \psi}{\partial \eta} \qquad \left(\begin{array}{l} \xi = {}^1/_2\,(t-\theta) \\ \eta = {}^1/_2\,(t+\theta) \end{array}\right) \tag{1.1}$$

Here $K_1(t)$ is the Chaplygin function; ξ and η are characteristic variables; t is the magnitude of the velocity vector; and θ the angle of the velocity vector with the x-axis.

If we take

$$K_1(t) = (n \tan mt)^4 \tag{1.2}$$

then for ϕ and ψ we obtain the usual solutions in the form

$$\varphi = n\left\{-m\,[f_1(\xi) + f_2(\eta)] + \tfrac{1}{2}\tan m(\xi+\eta)\,[f_1'(\xi) + f_2'(\eta)]\right\}$$

$$\psi = n^{-1}\left\{m\,[-f_1(\xi) + f_2(\eta)] + \tfrac{1}{2}\cot m(\xi+\eta)\,[-f_1'(\xi) + f_2'(\eta)]\right\} \tag{1.3}$$

where $f_1(\xi)$ and $f_2(\eta)$ are arbitrary functions, subject to boundary conditions, and n and m are arbitrary constants [8].

With condition (1.2) for the velocity v, density ρ and Mach number M we have

$$v(t) = \frac{Bn^2 \tan mt}{A\,(m \tan mt \sin t + \cos t) + (m \tan mt \cos t - \sin t)} \tag{1.4}$$

$$\rho(t) = \frac{A\,(m \tan mt \sin t + \cos t) + (m \tan mt \cos t - \sin t)}{n^2 \tan mt\,[A\,(\tan mt \sin t + m\cos t) + (\tan mt \cos t - m \sin t)]} \tag{1.5}$$

$$M(t) = \sqrt{1 + \rho^2 K_1} \tag{1.6}$$

where A and B complete the set of arbitrary constants. These may be suitably chosen, as well as the constants m, n, to obtain approximations of the degree desired [8].

From (1.4), (1.5) and (1.6) we establish that $\rho = 0$, $M = \infty$ and the maximum value of velocity is reached at $t = \pi/2m$.

We further note the positivity of the derivative dv/dt. This can be easily derived from the usual relation [9]

$$dv / dt = v / \rho \sqrt{K_1} \qquad (1.7)$$

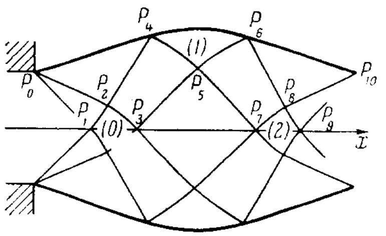

Fig. 1.

The flow under consideration is evidently symmetrical with respect to the x-axis, passing through the center of the exit in a direction parallel to the walls of the nozzle at the exit. Therefore, it is sufficient to limit the study to the upper half of the flow alone.

The set of characteristic regions, in which it is necessary to define the functions required, $f_1(\xi)$ and $f_2(\eta)$, are represented in the physical plane of the flow on Fig. 1 and in the plane of the variables t, θ on Fig. 2. Let corresponding points on Figs. 1 and 2 take on identical notations. In the plane t, θ these regions are bounded with segments of the lines $\theta = 0$ and $t = t_2$ and with segments of the characteristics

$$\begin{aligned} \xi = \xi_1 = {}^1/_2 (t_2 - \theta_2) = {}^1/_2 t_1, \quad \eta = \eta_1 = {}^1/_2 (t_2 + \theta_2) = {}^1/_2 t_3 \\ \xi = \xi_2 = {}^1/_2 (t_2 + \theta_2) = {}^1/_2 t_3, \quad \eta = \eta_2 = {}^1/_2 (t_2 - \theta_2) = {}^1/_2 t_1 \end{aligned} \qquad (1.8)$$

where t_1 is the value of the magnitude of t at the exit of the nozzle, t_2 is the constant value of t on the surface of the jet and in the region $P_0P_2P_4$, and $P_6P_8P_{10}$ and t_3 is the constant value of t in the region $P_3P_5P_7$. θ_2 is the angle characterizing the direction of the flow in the region $P_0P_2P_4$.

The characteristic regions in Figs. 1 and 2 are shown for a flow in which the parameters t_1 and t_2 correspond to the condition

$$2t_2 - t_1 < \pi / 2m \qquad (t_3 = 2t_2 - t_1) \qquad (1.9)$$

We shall restrict ourselves in the beginning to the examination of the flow corresponding to this condition (1.9).

2. Coming now to the solution of the problem, we turn first of all to the result of [10]. In this paper the question of the determination of the stream function on a cross-characteristic in a simple wave is examined. This leads to the following result: if on one of the cross-characteristics in the simple wave the function ψ^* is known, then the values of ψ on the second cross-characteristic can be expressed by the

formula

$$\psi = \psi^* + \mu K_1^{-1/4} \tag{2.1}$$

where μ is a quantity which remains constant on each characteristic; the continuous transition from one characteristic to the other carries with it a continuous change in μ. Moreover, if the simple wave is centered, then on the cross-characteristic

$$\psi = Q + \mu K_1^{-1/4} \tag{2.2}$$

where Q is the value of the stream function at the center.

In our problem, the pressure in the exterior region is below the pressure at the exit of the nozzle. Therefore, at the edge of the nozzle there arises a centered rarefaction wave. Letting $\psi = Q$ at the point P_0 and on the surface of the jet, and taking $\psi = 0$ on the axis of symmetry, we get, according to (2.2) and (1.2), the value of the stream function on the cross-characteristic P_1P_2 in the form

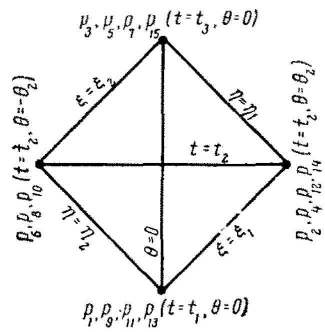

Fig. 2.

$$\psi_{P_1P_2} = Q\left(1 - \frac{\tan mt_1}{\tan m(\xi_1 + \eta)}\right) \tag{2.3}$$

Next, it is necessary to determine $f_1(\xi)$ and $f_2(\eta)$ in the region $P_1P_2P_3$ (region (0)) from the known values (2.3) of the function ψ on the characteristic P_1P_2 and the condition $\psi = 0$ on the axis of the symmetry. The solution of this boundary-value problem for the function $K_1(t)$ assumed in (1.2) can be found in [11]. In the problem considered here we get

$$f_1^{(0)}(\xi) = f(\xi), \quad f_2^{(0)}(\eta) = f(\eta) \quad \left(f(\xi) = -q\cos 2m(\xi + \eta_2), \quad q = \frac{nQ}{2m\cos^2 mt}\right) \tag{2.4}$$

We next go on to region $P_4P_5P_6$ (region (1)), bounded in the plane $t + \theta$ by segments of the characteristics $\xi = \xi_2$, $\eta = \eta_1$ and a segment of the line $t = t_2$. Here, it is necessary to satisfy two conditions: in the first place, on the cross-characteristic P_4P_5 the function ψ must attain the value determined from Formula (2.1); and, secondly, on the free surface P_4P_6 the relation $\psi = Q$ must hold.

The solution of the boundary-value problem with these given conditions on the characteristic and the free surface has also been given in [11].

According to the result of that paper, it is natural to represent the desired solution in the form

$$f_1^{(1)}(\xi) = f(\xi)$$
$$f_2^{(1)}(\eta) = -f(t_2-\eta) + 2k_2 e^{-k_2\eta}\int_{\eta_1}^{\eta} f(t_2-\eta)\, e^{k_2\eta}\, d\eta + C_1 e^{-k_2\eta} + C_2 \qquad (2.5)$$

where $k_2 = 2m \tan (mt_2)$ and C_1 and C_2 are constants which are determined from the boundary conditions.

Satisfying the condition on the free surface $\psi = Q$ and then the condition on the characteristic P_4P_5, we find

$$C_2 = 2q\cos^2 mt_1, \qquad C_1 = -qe^{k_2\eta_1}(\cos 2mt_1 + 2\cos^2 mt_1 + \cos 2mt_2) \qquad (2.6)$$

As a result of the integration the function $f_2^{(1)}(\eta)$ can be represented in the form

$$f_2^{(1)}(\eta) = q\{\cos 2m\ (\eta_2-\eta) -$$
$$- 4e^{k_2(\eta_1-\eta)}\cos mt_1 \cos mt_2 \cos m(t_2-t_1) + 2\cos^2 mt_1\} \qquad (2.7)$$

In the region $P_7P_8P_9$ (region (2)), taking into account symmetry with respect to the x-axis, we easily obtain

$$f_1^{(2)}(\xi) = f_2^{(1)}(\xi), \qquad f_2^{(2)}(\eta) = f_2^{(1)}(\eta) \qquad (2.8)$$

The values of ψ on the cross-characteristics P_5P_6 and P_7P_8 are obtained by the relation (2.1).

It will be shown below that the solution for simple waves given on the cross-characteristic P_8P_9 always contains a singularity, the limit line, the existence of which indicates the impossibility of further extending a continuous potential flow in the jet. However, in the plane $t\theta$ no such singularity, as is known, appears. Therefore, it is possible formally to continue the solution, starting with the condition $\psi(t, 0) = 0$, $\psi(t_2, \theta) = Q$ and using the relation (2.1) for the function ψ.

In the region $P_{10}P_{11}P_{12}$ (region (3)) by analogy with (2.5) we obtain

$$f_1^{(3)}(\xi) = f_1^{(2)}(\xi), \qquad f_2^{(3)}(\eta) = f(\eta) + C_3 e^{-k_2\eta} + C_4 \qquad (2.9)$$

where

$$C_4 = 4q\cos^2 mt_1, \qquad C_3 = -4qe^{k_2\eta_1}\cos mt_1 \cos mt_2 \cos m(t_2-t_1) \qquad (2.10)$$

In region $P_{13}P_{14}P_{15}$ (region (4)) the desired functions $f_1^{(4)}(\xi)$ and

$f_2^{(4)}(\eta)$ are determined easily in the same way as for the functions (2.8) in region $P_7P_8P_9$

$$f_1^{(4)}(\xi) = f(\xi) + C_3 e^{-k_2 \xi}, \qquad f_2^{(4)}(\eta) = f(\eta) + C_3 e^{-k_2 \eta} \tag{2.11}$$

Comparing (2.4) and (2.11), one finds a solution is obtained which cannot be periodic since $C_3 \neq 0$.

3. We now show that in the jet limit lines always occur. For that purpose we consider the solution in a region, abutting the characteristics P_8P_9 and P_8P_{10}, where the appearance of the limit line seems to be most probable.

The flow in that special region is represented by simple waves so that the usual method of investigation, which consists of studying the Jacobian $\Delta = \partial(\phi, \psi)/\partial(v, \theta)$, is not acceptable here. In this case it is necessary to make use of another condition for the existence of the limit line, which may be obtained for simple waves from [12]. It consists of satisfying the equation

$$\psi_\xi'(\xi, \eta) = 0 \tag{3.1}$$

for points of the plane ξ, η, the images of which in the physical plane of the gas flow can be found on the limit line.

In order that we may verify that condition (3.1) is fulfilled we will find the values of the function $\psi_\xi'(\xi, \eta)$ for the points $P_9(\xi = \xi_1, \eta = \eta_2)$, $P_{10}(\xi = \xi_2, \eta = \eta_2)$ and $P_{11}(\xi = \xi_1, \eta = \eta_2)$. If it appears that two of the defined quantities have different signs, then the fulfilling of condition (3.1) and, consequently, the existence of a limit line will be guaranteed.

On the characteristics P_8P_9 and $P_{10}P_{11}$ the function $\psi_\xi'(\xi, \eta)$ does not have a discontinuity. Therefore, for the determination of the value of this function at P_9 use is made of (2.8), and to determine the value at P_{10} and P_{11} solution (2.9) is used.

The expression for $\psi_\xi'(\xi)$, on the basis of (1.3), can be represented in the convenient form

$$\frac{\partial \psi}{\partial \xi} = -\frac{1}{2n \sin^2 mt}\left[m f_2'(\eta) - m \cos 2mt f_1'(\xi) + \frac{\sin 2mt}{2} f_1''(\xi)\right] \tag{3.2}$$

The expression for the derivative at the point P_9 has the form

$$\left(\frac{\partial \psi}{\partial \xi}\right)_{P_9} = \frac{2m^2 q}{n} \cot mt_1 \left[1 + 2e^{k_2(\eta_1 - \eta_2)} \tan mt_2 \sin 2m(t_2 - t_1)\right] \tag{3.3}$$

From this it is evident that the function considered is always positive at P_9 ($t_2 > t_1$). At the point P_{11} we have

$$\left(\frac{\partial\psi}{\partial\xi}\right)_{P_{11}} = \frac{4m^2q}{n} e^{k_2(\eta_1-\eta_2)} \cot mt_1 \tan mt_2 \sin 2m(t_2 - t_1) \tag{3.4}$$

This expression takes on only positive values for the same reason as in (3.3). At the point P_{10} we have

$$\left(\frac{\partial\psi}{\partial\xi}\right)_{P_{10}} = -\frac{4m^2q}{n} \frac{\cos mt_1 \cos m(t_2 - t_1)}{\sin mt_2} [e^{k_2(\eta_1-\eta_2)} - 1] \tag{3.5}$$

Unlike the expressions in (3.3) and (3.4), the derivative at P_{10} takes on essentially negative values, because $\eta_1 > \eta_2$.

The analysis which has been carried out shows that the condition for the existence of the limit line in the simple wave on the cross-characteristic P_8P_9 is satisfied and it follows that continuous potential flow at some finite distance from the end of the nozzle is impossible. This conclusion is correct both for small pressure drops and for larger ones. Thus, the condition in [5] on the size of the pressure drops is removed. It appeared only because of the method of proof used in [5].

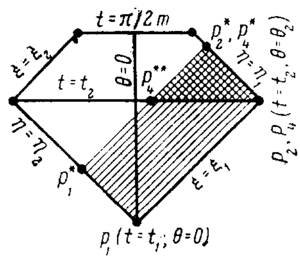

Fig. 3.

Note, however, that for large pressure drops the continuous solutions obtained can lose their physical meaning even earlier. As an example we consider the solution for the region $P_4P_5P_6$ and we show that for a sufficiently large value of t_2 on some line in this region there will necessarily be satisfied the condition for the existence of a limit line, $\Delta = 0$.

With the help of the basic equations [9]

$$\frac{\partial\varphi}{\partial\theta} = \sqrt{K_1}\,\frac{\partial\psi}{\partial t}, \qquad \frac{\partial\varphi}{\partial t} = \sqrt{K_1}\,\frac{\partial\psi}{\partial\theta} \tag{3.6}$$

which are equivalent to the systems (1.1), the Jacobian Δ can be presented in the form

$$\Delta = \left[K_1^{1/2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 - K_1^{-1/2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right]\frac{dt}{dv} \tag{3.7}$$

From this it follows that $\Delta \leqslant 0$ on P_4P_6, where $\psi = \text{const}$.

Next, using the solution (2.5), we find the value of the Jacobian at the point P_5. For this we transform the expression for Δ to the convenient form

$$\Delta = \frac{1}{\sin^2 2mt}\left[mf_1'(\xi) - m\cos 2mtf_2'(\eta) + \frac{\sin 2mt}{2} f_2''(\eta)\right]\times$$
$$\times\left[mf_2'(\eta) - m\cos 2mtf_1'(\xi) + \frac{\sin 2mt}{2} f_1''(\xi)\right]\frac{dt}{dv} \quad (3.8)$$

After substituting (2.5) into (3.8) we get

$$\Delta_{P_5} = \frac{4m^4q^2}{\sin^2 2mt_3}[\sin 2mt_2 - \cos 2mt_3(\sin 2mt_1 + \sin 2mt_2) - \sin 2mt_3(\cos 2mt_1 +$$
$$+ 2\sin 2mt_1 \operatorname{tg} mt_2 + 2\sin^2 mt_2)][\sin 2mt_1 + \sin 2mt_2 + \sin 2m(t_3 - t_2)]\frac{dt}{dv} \quad (3.9)$$

If we fix t_1 and we increase t_2, then t_3 also increases. For $t_3 \to \pi/2m$ the last member of the first square brackets (3.9) becomes small, and the second member becomes negative. It follows that for t_3 sufficiently close to $\pi/2m$ the Jacobian at the point P_5 has a positive value. Because Δ in the region $P_4P_5P_6$ is a continuous function on some line it changes sign, and that means that the condition for the existence of a limit line in the considered region is fulfilled.

4. If the parameters t_1 and t_2 do not satisfy the inequality (1.9) and

$$t_2 \geqslant \frac{\pi}{4m} + \frac{t_1}{2} \qquad \left(t_1 < \frac{\pi}{2m}\right) \quad (4.1)$$

then the limit line is always formed to the right of the characteristic P_4P_5.

In order to prove this we consider the region $P_4P_4{}^*P_4{}^{**}$, bounded by the segment of the straight line $P_4P_4{}^{**}(t = t_2)$ and the segments of the characteristic $P_4P_4{}^*$ and $P_4{}^*P_4{}^{**}$, where the characteristic $P_4{}^*P_4{}^{**}$ is drawn in such a way that at $P_4{}^*$ the value of the variable t is sufficiently close to $\pi/2m$ (Fig. 3).

The desired functions, which determine the solution in the divided region, will be the functions (2.5). We shall obtain that result if we first solve the boundary problem in the region $P_1P_2P_2{}^*P_1{}^*$ from that given by (2.3), and afterwards we determine the values of ψ on the cross-characteristic in the simple wave.

Using (2.5), one can find the value of the Jacobian Δ at the point $P_1{}^*$. If t is in $P_4{}^*$ sufficiently close to $\pi/2m$ then it is easy to deduce that at that point Δ will be positive. Taking into account that on $P_4P_4{}^{**}$ (the free surface) Δ has the opposite sign ($\Delta < 0$), it is easy to see that the condition $\Delta = 0$ is satisfied on some line.

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